A SIMPLE PROOF OF HART'S THEOREM.

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The theorem which forms the subject of the present note was discovered by Sir Andrew Hart[1] and may be stated as follows:

If a triangle be formed by the arcs of three circles, the inscribed and the three escribed circles are all tangent to a new circle or line.

This theorem is to be found in numerous works on modern elementary geometry and has been proved also in countless shorter articles. All of the proofs with which I am familiar, at least all of those which are elementary in character, are essentially alike and depend upon Casey's criterion for four circles tangent to a fifth. If the circles be c_1, c_2, c_3, c_4, a necessary and sufficient condition that they be tangent to a circle or line is that there should exist among their common tangents a relation

$$t_{12}t_{34} \pm t_{13}t_{42} \pm t_{14}t_{23} = 0,$$

where t_{ij} means the length of a common tangent to c_i and c_j.[2] Hart's theorem comes very quickly by four successive applications of this relation. The trouble is that the amount of confidence that one is willing to repose in this criterion varies inversely as the amount of care with which one regards it. The difficulties are as follows:

(*a*) The proof that this equation affords a sufficient condition that four circles should be tangent to a fifth is rather intricate.[3]

(*b*) The condition, as a theorem in elementary geometry, is not necessary. We may have four circles tangent to a fifth where one of the four surrounds the other three. Here, three of the common tangents simply do not exist in the universe of discourse.

(*c*) Some of the common tangents may be direct and some transverse, and exact statement covering all cases correctly is cumbersome.

(*d*) It is a tedious task to look after all of the + and − signs. The usual method is to write them all + and trust in Providence.

It is for these reasons that it seems to me worth while to give another proof based upon different considerations. Our method consists simply in throwing Hart's theorem back upon the theorem of Feuerbach, and assuming that the reader is familiar with one, at least, of the many ordinary proofs of the latter.[4] Feuerbach's theorem states that the inscribed and escribed circles of a triangle are tangent to a fourth circle, namely, the nine-point circle.

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[1] "On the Extension of Terquem's Theorem," *Quarterly Journal*, Vol. IV, 1860.

[2] Conf. Casey's *Sequel to Euclid*, second ed., London, 1881, p. 101.

[3] Conf. Lachlan, *Modern Pure Geometry*, London, 1893, p. 244. Casey avoids the difficulty by assuming that a necessary condition must be sufficient.

[4] Certain mathematicians seem to consider that the proving of Feuerbach's theorem constitutes a separate branch of mathematics. Conf. Sawayama, "Nouvelles démonstrations d'un théorème relatif au cercle de neuf points," *L'Enseignement mathématique*, Vol. XIII, 1911. This paper contains nine new proofs.

Proof. The circles constituting the triangle shall be $c_1c_2c_3$, the inscribed circle c, while the escribed circles need not be named. Let c' be the circle tangent to c_1, c_2, c_3 which couples with c, *i. e.*, has either exactly the same contacts with all three, or exactly the opposite contacts. If c' reduce to a point, we may at once invert c_1, c_2, c_3 into three lines, and we have reached Feuerbach's theorem. If c' be not a point or a straight line (a case which we may avoid by a preliminary inversion) let N be a point on the axis of this circle at a radius's distance from the center. We take a sphere of inversion with N as center, passing through c'. The plane will invert into that sphere which has N for north pole and c' for equator, and our problem consists in proving Hart's theorem for the sphere, where the given circles touch the equator. We next make the simple transformation which consists in replacing each great circle by one of its poles, say that which lies in the northern hemisphere. This transformation carries a circle into a circle, and tangent circles into tangent circles. Hence we have merely to prove Hart's theorem on a sphere where the three given circles pass through the north pole. But if we repeat our previous inversion, the three circles through the north pole become three straight lines in the plane, and we are carried back to Feuerbach's theorem once more. Our proof of the dependence of the one theorem upon the other is, thus, complete.

A TRIBUTE TO JOHN HOWARD VAN AMRINGE.

In the death of Professor John Howard Van Amringe September 10, 1915, in his eighty-first year, there passed away from Columbia College one of the greatest men, and without doubt the most beloved man, ever connected with that venerable institution.

Born of Dutch parentage in Philadelphia in 1836 and prepared for college mainly by his father, he entered Yale in 1854. At the end of his sophomore year he transferred to Columbia College and was graduated from this institution with the degree of A.B. in 1860, at the age of twenty-four. So brilliant and many-sided were his native powers and his attainments that even before graduation he had been tendered an instructorship in no fewer than five widely diverse departments: Greek, Latin, history, chemistry, mathematics. He might with equal propriety have been invited into the department of English, had there been such a department at that time, for his extant writings, including many published addresses, show that he had a remarkable control over the resources of English speech.

He chose mathematics and this subject he taught till his resignation from Columbia University, October, 1909, after fifty years of service as teacher and many years of service as dean of the college. Why he chose this science I do not know, but, as President Butler has felicitously said, "There was something curiously appropriate in his choice of mathematics as the agency of his academic influence, for there was in it that rigor of demonstration and that accuracy of statement which marched so well with his sturdy uprightness, his straight